Outline

1. Iterative Improvement (Revisited)

2. ‘Simple’ SLS Methods

3. Hybrid SLS Methods

4. Population-based SLS Methods
Iterative Improvement (II):

determine initial candidate solution $s$

While $s$ is not a local optimum:

1. choose a neighbour $s'$ of $s$ such that $g(s') < g(s)$
2. $s := s'$

Main Problem:
Stagnation in local optima of evaluation function $g$. 
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- **Idea**: Reduce size of neighbourhoods by excluding neighbours that are likely (or guaranteed) not to yield improvements in $g$.
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Example: Candidate lists for the TSP

- **Intuition**: High-quality solutions likely include short edges.
- **Candidate list** of vertex $v$: list of $v$'s nearest neighbours (limited number), sorted according to increasing edge weights.
- Search steps (e.g., 2-exchange moves) always involve edges to elements of candidate lists.
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- **Best Improvement** (aka gradient descent, greedy hill-climbing): Choose maximally improving neighbour, i.e., randomly select from $l^*(s) := \{s' \in N(s) \mid g(s') = g^*\}$, where $g^* := \min\{g(s') \mid s' \in N(s)\}$.

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- **Given:** TSP instance \( G \) with vertices \( v_1, v_2, \ldots, v_n \).
- **Search space:** Hamiltonian cycles in \( G \);
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- **Initialisation:**
  search position := fixed canonical path \((v_1, v_2, \ldots, v_n, v_1)\)
  \( P := \text{random permutation of } \{1,2, \ldots, n\} \)

- **Search steps:** determined using first improvement w.r.t. \( g(p) = \text{weight of path } p \), evaluating neighbours in order of \( P \) (does not change throughout search)

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Empirical performance evaluation:

- Perform 1000 runs of algorithm on benchmark instance pcb3038.
- Record relative solution quality ( = percentage deviation from known optimum) of final tour obtained in each run.
- Plot cumulative distribution function of relative solution quality over all runs.
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Example: Random-order first improvement for the TSP (3)

Result: Substantial variability in solution quality between runs.
Variable Neighbourhood Descent

- **Recall**: Local minima are relative to neighbourhood relation.
- **Key idea**: To escape from local minimum of given neighbourhood relation, switch to different neighbourhood relation.
- Use $k$ neighbourhood relations $N_1, \ldots, N_k$, (typically) ordered according to increasing neighbourhood size.
- Always use smallest neighbourhood that facilitates improving steps.
- Upon termination, candidate solution is locally optimal w.r.t. all neighbourhoods.
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Variable Neighbourhood Descent (VND):

determine initial candidate solution $s$

$i := 1$

Repeat:

| choose a most improving neighbour $s'$ of $s$ in $N_i$
| If $g(s') < g(s)$:
| $s := s'$
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| Else:
| $i := i + 1$

Until $i > k$
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- VND often performs substantially better than simple II or II in large neighbourhoods [Hansen and Mladenović, 1999]
- Many variants exist that switch between neighbourhoods in different ways.
- More general framework for SLS algorithms that switch between multiple neighbourhoods: Variable Neighbourhood Search (VNS) [Mladenović and Hansen, 1997].
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Variable Depth Search

- **Key idea:** Complex steps in large neighbourhoods = variable-length sequences of simple steps in small neighbourhood.
  
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$\hat{t} := s$

While $s$ is not locally optimal:

Repeat:

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Until construction of complex step has been completed

$s := \hat{t}$
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Example: The Lin-Kernighan (LK) Algorithm for the TSP (1)

Complex search steps correspond to sequences of 2-exchange steps and are constructed from sequences of Hamiltonian paths (= paths that visit every node in given graph exactly once).

\( \delta \)-path: Hamiltonian path \( p \) + 1 edge connecting one end of \( p \) to interior node of \( p \) (‘lasso’ structure):
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- **Note**: Hamiltonian path can be completed into Hamiltonian cycle by adding edge \((v', u)\):
  
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1. start with current candidate solution (Hamiltonian cycle) \( s \); set \( t^* := s \); set \( p := s \)
2. obtain \( \delta \)-path \( p' \) by replacing one edge in \( p \)
3. consider Hamiltonian cycle \( t \) obtained from \( p \) by (uniquely) defined edge exchange
4. if \( w(t) < w(t^*) \) then set \( t^* := t \); \( p := p' \); go to step 2
5. else accept \( t^* \) as new current candidate solution \( s \)

**Note:** This can be interpreted as sequence of 1-exchange steps that alternate between \( \delta \)-paths and Hamiltonian cycles.
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Additional mechanisms used by LK algorithm:

- *Tabu restriction:* Any edge that has been added cannot be removed and any edge that has been removed cannot be added in the same LK step.

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Note:

Variable depth search algorithms have been very successful for other problems, including:

- the Graph Partitioning Problem [Kernigan and Lin, 1970];
- the Unconstrained Binary Quadratic Programming Problem [Merz and Freisleben, 2002];
- the Generalised Assignment Problem [Yagiura et al., 1999].
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- Iterative improvement method based on building complex search steps from combinations of simple search steps.
- Simple search steps constituting any given complex step are required to be *mutually independent*, i.e., do not interfere with each other w.r.t. effect on evaluation function and feasibility of candidate solutions.

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- Iterative improvement method based on building complex search steps from combinations of simple search steps.
- Simple search steps constituting any given complex step are required to be \textit{mutually independent}, i.e., do not interfere with each other w.r.t. effect on evaluation function and feasibility of candidate solutions.

\textit{Example:} Independent 2-exchange steps for the TSP:

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{example.png}
\end{figure}

\textit{Therefore:} Overall effect of complex search step = sum of effects of constituting simple steps; complex search steps maintain feasibility of candidate solutions.
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- **Key idea:** Efficiently find optimal combination of mutually independent simple search steps using *Dynamic Programming*.

- Successful applications to various combinatorial optimisation problems, including:
  - the TSP and the Linear Ordering Problem [Congram, 2000]
  - the Single Machine Total Weighted Tardiness Problem (scheduling) [Congram *et al.*, 2002]
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‘Simple’ SLS Methods

Goal:
Effectively escape from local minima of given evaluation function.

General approach:
For fixed neighbourhood, use step function that permits worsening search steps.

Specific methods:
- Randomised Iterative Improvement
- Probabilistic Iterative Improvement
- Simulated Annealing
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Randomised Iterative Improvement

**Key idea:** In each search step, with a fixed probability perform an uninformed random walk step instead of an iterative improvement step.

Randomised Iterative Improvement (RII):

determine initial candidate solution $s$

While termination condition is not satisfied:

- With probability $wp$:
  - choose a neighbour $s'$ of $s$ uniformly at random
- Otherwise:
  - choose a neighbour $s'$ of $s$ such that $g(s') < g(s)$ or,
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Note:

- No need to terminate search when local minimum is encountered
  
  *Instead:* Bound number of search steps or CPU time from beginning of search or after last improvement.

- Probabilistic mechanism permits arbitrary long sequences of random walk steps
  
  *Therefore:* When run sufficiently long, RII is guaranteed to find (optimal) solution to any problem instance with arbitrarily high probability.

- A variant of RII has successfully been applied to SAT (GWSAT algorithm), but generally, RII is often outperformed by more complex SLS methods.
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Example: Randomised Iterative Best Improvement for SAT

procedure GUWSAT($F$, $wp$, $maxSteps$)
    input: propositional formula $F$, probability $wp$, integer $maxSteps$
    output: model of $F$ or $\emptyset$

    choose assignment $a$ of truth values to all variables in $F$
    uniformly at random;
    $steps := 0$;
    while not ($a$ satisfies $F$) and ($steps < maxSteps$) do
        with probability $wp$ do
            select $x$ uniformly at random from set of all variables in $F$;
        otherwise
            select $x$ uniformly at random from \{$x' \mid x'$ is a variable in $F$ and changing value of $x'$ in $a$ max. decreases number of unsat. clauses$\};
        end
        change value of $x$ in $a$;
        $steps := steps + 1$;
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    if $a$ satisfies $F$ then return $a$
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Probabilistic Iterative Improvement

**Key idea:** Accept worsening steps with probability that depends on respective deterioration in evaluation function value: bigger deterioration $\Rightarrow$ smaller probability

**Realisation:**

- Function $p(g, s)$: determines probability distribution over neighbours of $s$ based on their values under evaluation function $g$.
- Let $\text{step}(s)(s') := p(g, s)(s')$.

**Note:**

- Behaviour of PII crucially depends on choice of $p$.
- II and RII are special cases of PII.
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Example: Metropolis PII for the TSP (1)

- **Search space:** set of all Hamiltonian cycles in given graph $G$.

- **Solution set:** same as search space (i.e., all candidate solutions are considered feasible).

- **Neighbourhood relation:** reflexive variant of 2-exchange neighbourhood relation (includes $s$ in $N(s)$, i.e., allows for steps that do not change search position).
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- **Initialisation:** pick Hamiltonian cycle uniformly at random.

- **Step function:** implemented as 2-stage process:
  1. select neighbour \( s' \in N(s) \) uniformly at random;
  2. accept as new search position with probability:

\[
p(T, s, s') := \begin{cases} 
1 & \text{if } f(s') \leq f(s) \\
\exp\left(\frac{f(s) - f(s')}{T}\right) & \text{otherwise}
\end{cases}
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( *Metropolis condition*), where temperature parameter \( T \) controls likelihood of accepting worsening steps.

- **Termination:** upon exceeding given bound on run-time.
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Simulated Annealing

**Key idea:** Vary temperature parameter, i.e., probability of accepting worsening moves, in Probabilistic Iterative Improvement according to *annealing schedule* (aka *cooling schedule*).

*Inspired by physical annealing process:*

- candidate solutions $\simeq$ states of physical system
- evaluation function $\simeq$ thermodynamic energy
- globally optimal solutions $\simeq$ ground states
- parameter $T \simeq$ physical temperature

*Note:* In physical process (e.g., annealing of metals), perfect ground states are achieved by very slow lowering of temperature.
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Simulated Annealing (SA):

determine initial candidate solution \( s \)
set initial temperature \( T \) according to \textit{annealing schedule}

While termination condition is not satisfied:

| probabilistically choose a neighbour \( s' \) of \( s \) using \textit{proposal mechanism} |
| If \( s' \) satisfies probabilistic \textit{acceptance criterion} (depending on \( T \)):
| \( s := s' \) |
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- 2-stage step function based on
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Annealing schedule (function mapping run-time $t$ onto temperature $T(t)$):

- initial temperature $T_0$
  (may depend on properties of given problem instance)
- temperature update scheme
  (e.g., geometric cooling: $T := \alpha \cdot T$)
- number of search steps to be performed at each temperature
  (often multiple of neighbourhood size)

Termination predicate: often based on *acceptance ratio*,
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Example: Simulated Annealing for the TSP

Extension of previous PII algorithm for the TSP, with

- **proposal mechanism**: uniform random choice from 2-exchange neighbourhood;

- **acceptance criterion**: Metropolis condition (always accept improving steps, accept worsening steps with probability \( \exp \left[ \frac{(f(s) - f(s'))}{T} \right] \));

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‘Convergence’ result for SA:

Under certain conditions (extremely slow cooling), any sufficiently long trajectory of SA is guaranteed to end in an optimal solution [Geman and Geman, 1984; Hajek, 1998].

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Key idea: Use aspects of search history (memory) to escape from local minima.

Simple Tabu Search:

- Associate *tabu attributes* with candidate solutions or solution components.
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determine initial candidate solution $s$

While *termination criterion* is not satisfied:

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- Non-tabu search positions in $N(s)$ are called *admissible neighbours of $s$.*

- After a search step, the current search position or the solution components just added/removed from it are declared *tabu* for a fixed number of subsequent search steps (*tabu tenure*).

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- **Search space:** set of all truth assignments for propositional variables in given CNF formula $F$.
- **Solution set:** models of $F$.
- Use 1-flip **neighbourhood relation**, i.e., two truth assignments are neighbours iff they differ in the truth value assigned to one variable.
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- $tt$ too low $\Rightarrow$ search stagnates due to inability to escape from local minima;
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**Advanced TS methods:**

- **Robust Tabu Search** [Taillard, 1991]: repeatedly choose $tt$ from given interval; *also:* force specific steps that have not been made for a long time.

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- **Key Idea:** Modify the evaluation function whenever a local optimum is encountered in such a way that further improvement steps become possible.

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*initialise penalties*

While *termination criterion* is not satisfied:

- compute *modified evaluation function* $g'$ from $g$ based on *penalties*

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Dynamic Local Search (continued)

- **Modified evaluation function:**

\[ g'(\pi, s) := g(\pi, s) + \sum_{i \in SC(\pi', s)} \text{penalty}(i), \]

where \( SC(\pi', s) = \) set of solution components of problem instance \( \pi' \) used in candidate solution \( s \).

- **Penalty initialisation:** For all \( i \): \( \text{penalty}(i) := 0 \).

- **Penalty update** in local minimum \( s \): Typically involves \( \text{penalty increase} \) of some or all solution components of \( s \); often also occasional \( \text{penalty decrease} \) or \( \text{penalty smoothing} \).

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Solution components required for (optimal) solution may also be present in many local minima.

Possible solutions:

A: Occasional decreases/smoothing of penalties.
B: Only increase penalties of solution components that are least likely to occur in (optimal) solutions.

Implementation of B:
[Voudouris and Tsang, 1995]

Only increase penalties of solution components $i$ with maximal utility:

$$\text{util}(s', i) := \frac{f_i(\pi, s')}{1 + \text{penalty}(i)}$$

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Example: Guided Local Search (GLS) for the TSP
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- **Given:** TSP instance \( G \)
- **Search space:** Hamiltonian cycles in \( G \) with \( n \) vertices; use standard 2-exchange neighbourhood;
  solution components = edges of \( G \);
  \( f(G, p) := w(p); f_e(G, p) := w(e); \)

- **Penalty initialisation:** Set all edge penalties to zero.

- **Subsidiary local search:** Iterative First Improvement.

- **Penalty update:** Increment penalties for all edges with maximal utility by
  \[ \lambda := 0.3 \cdot \frac{w(s_{2-opt})}{n} \]

  where \( s_{2-opt} \) = 2-optimal tour.
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  \[ f(G, p) := w(p); \ f_e(G, p) := w(e); \]

- **Penalty initialisation:** Set all edge penalties to zero.

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- **Penalty update:** Increment penalties for all edges with maximal utility by
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Earlier, closely related methods:

- Breakout Method [Morris, 1993]
- GENET [Davenport et al., 1994]
- Clause weighting methods for SAT [Selman and Kautz, 1993; Cha and Iwama, 1996; Frank, 1997]

Dynamic local search algorithms are state of the art for many problems, including:

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Hybrid SLS Methods

Combination of ‘simple’ SLS methods often yields substantial performance improvements.

Simple examples:

- Commonly used restart mechanisms can be seen as hybridisations with Uninformed Random Picking
- Iterative Improvement + Uninformed Random Walk = Randomised Iterative Improvement
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Iterated Local Search

**Key Idea:** Use two types of SLS steps:

- *subsidiary local search* steps for reaching local optima as efficiently as possible (intensification)
- *perturbation steps* for effectively escaping from local optima (diversification).

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Iterated Local Search (ILS):

determine initial candidate solution $s$
perform *subsidiary local search* on $s$

While termination criterion is not satisfied:

$r := s$
perform *perturbation* on $s$
perform *subsidiary local search* on $s$

based on *acceptance criterion*,
keep $s$ or revert to $s := r$
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Note:

- *Subsidiary local search* results in a local minimum.
- ILS trajectories can be seen as walks in the space of local minima of the given evaluation function.
- *Perturbation phase* and *acceptance criterion* may use aspects of *search history* (i.e., limited memory).
- In a high-performance ILS algorithm, *subsidiary local search*, *perturbation mechanism* and *acceptance criterion* need to complement each other well.
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- More effective subsidiary local search procedures lead to better ILS performance.

  *Example:* 2-opt vs 3-opt vs LK for TSP.

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- Needs to be chosen such that its effect *cannot* be easily undone by subsequent local search phase. (Often achieved by search steps larger neighbourhood.)

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- Weak perturbation $\Rightarrow$ short subsequent local search phase; \textit{but}: risk of revisiting current local minimum.

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- Always accept the *better* of the two candidate solutions
  ⇒ ILS performs Iterative Improvement in the space of local optima reached by subsidiary local search.

- Always accept the *more recent* of the two candidate solutions
  ⇒ ILS performs random walk in the space of local optima reached by subsidiary local search.

- Intermediate behaviour: select between the two candidate solutions based on the *Metropolis criterion* (e.g., used in *Large Step Markov Chains* [Martin et al., 1991].

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  ![Diagram of the double-bridge move]

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- This ILS algorithm for the TSP is known as *Iterated Lin-Kernighan (ILK) Algorithm*.
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Iterated local search algorithms . . .

- are typically rather easy to implement (given existing implementation of subsidiary simple SLS algorithms);
- achieve state-of-the-art performance on many combinatorial problems, including the TSP.

There are many SLS approaches that are closely related to ILS, including:

- Large Step Markov Chains [Martin et al., 1991]
- Chained Local Search [Martin and Otto, 1996]
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**Greedy Randomised Adaptive Search Procedures**

**Key Idea:** Combine randomised constructive search with subsequent perturbative local search.

**Motivation:**

- Candidate solutions obtained from construction heuristics can often be substantially improved by perturbative local search.
- Perturbative local search methods typically often require substantially fewer steps to reach high-quality solutions when initialised using greedy constructive search rather than random picking.
- By iterating cycles of constructive + perturbative search, further performance improvements can be achieved.
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Greedy Randomised “Adaptive” Search Procedure (GRASP):

While termination criterion is not satisfied:

- generate candidate solution $s$ using subsidiary greedy randomised constructive search
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Randomisation in constructive search ensures that a large number of good starting points for subsidiary local search is obtained.
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Restricted candidate lists (RCLs)

- Each step of constructive search adds a solution component selected uniformly at random from a restricted candidate list (RCL).

- RCLs are constructed in each step using a heuristic function $h$.

- RCLs based on cardinality restriction comprise the $k$ best-ranked solution components. ($k$ is a parameter of the algorithm.)

- RCLs based on value restriction comprise all solution components $l$ for which $h(l) \leq h_{\text{min}} + \alpha \cdot (h_{\text{max}} - h_{\text{min}})$, where $h_{\text{min}} =$ minimal value of $h$ and $h_{\text{max}} =$ maximal value of $h$ for any $l$. ($\alpha$ is a parameter of the algorithm.)
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- **Given:** CNF formula $F$ over variables $x_1, \ldots, x_n$

- **Subsidiary constructive search:**
  - start from empty variable assignment
  - in each step, add one atomic assignment (i.e., assignment of a truth value to a currently unassigned variable)
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Key Idea: Alternate construction and perturbative local search phases as in GRASP, exploiting experience gained during the search process.

Realisation:

▷ Associate weights with possible decisions made during constructive search.

▷ Initialise all weights to some small value $\tau_0$ at beginning of search process.

▷ After every cycle (= constructive + perturbative local search phase), update weights based on solution quality and solution components of current candidate solution.
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- It is often useful to design solution component selection in constructive search such that any solution component may be chosen (at least with some small probability) irrespective of its weight and heuristic value.
Subsidiary perturbative local search:

- As in GRASP, perturbative local search phase is typically important for achieving good performance.
- Can be based on Iterative Improvement or more advanced SLS method (the latter often results in better performance).
- Tradeoff between computation time used in construction phase vs local search phase (typically optimised empirically, depends on problem domain).
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(Based on Ant System for the TSP [Dorigo et al., 1991].)

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Adaptive Iterated Construction Search ...

- models recent variants of constructive search, including:
  - stochastic tree search [Bresina, 1996],
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Population-based SLS Methods

SLS methods discussed so far manipulate one candidate solution of given problem instance in each search step.

**Straightforward extension:** Use *population* (i.e., set) of candidate solutions instead.

**Note:**

- The use of populations provides a generic way to achieve search diversification.
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Key idea: Can be seen as population-based extension of AICS where population of agents – *(artificial)*ants – communicate via common memory – *(simulated)*pheromone trails.

Inspired by foraging behaviour of real ants:

- Ants often communicate via chemicals known as *pheromones*, which are deposited on the ground in the form of trails. (This is a form of *stigmergy*: indirect communication via manipulation of a common environment.)

- Pheromone trails provide the basis for (stochastic) trail-following behaviour underlying, e.g., the collective ability to find shortest paths between a food source and the nest.
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Application to combinatorial problems:
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- Ants iteratively construct candidate solutions.
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Ant Colony Optimisation (ACO):

*initialise pheromone trails*

While termination criterion is not satisfied:

- generate population $sp$ of candidate solutions using *subsidiary randomised constructive search*
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- In each cycle, each ant creates one candidate solution using a *constructive search procedure*.
- *Subsidiary local search* is applied to individual candidate solutions. (Some ACO algorithms do not use a subsidiary local search procedure.)
- All *pheromone trails* are initialised to the same value, $\tau_0$.
- *Pheromone update* typically comprises uniform decrease of all trail levels (*evaporation*) and increase of some trail levels based on candidate solutions obtained from construction + local search.
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*Motivation*: Edges belonging to highest-quality candidate solutions and/or that have been used by many ants should be preferably used in subsequent constructions.
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Example: A simple ACO algorithm for the TSP (3)

- **Termination**: After fixed number of cycles
  \((= \text{construction} + \text{local search phases})\).

Note:

- Ants can be seen as walking along edges of given graph
  (using memory to ensure their tours correspond to
  Hamiltonian cycles) and depositing pheromone to reinforce
  edges of tours.

- Original Ant System did not include subsidiary local search
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Enhancements:

- use of look-ahead in construction phase;
- pheromone updates during construction phase;
- bounds on range and smoothing of pheromone levels.

Advanced ACO methods:

- Ant Colony System [Dorigo and Gambardella, 1997]
- $\text{MAX} - \text{MIN}$ Ant System [Stützle and Hoos, 1997; 2000]
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Ant Colony Optimisation . . .

- has been applied very successfully to a wide range of combinatorial problems, including
  - the Open Shop Scheduling Problem,
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Note:

A general algorithmic framework for solving static and dynamic combinatorial problems using ACO techniques is provided by the *ACO metaheuristic* [Dorigo and Di Caro, 1999; Dorigo et al., 1999].

For further details on Ant Colony Optimisation, see the book by Dorigo and Stützle [2004].
**Evolutionary Algorithms**

**Key idea:** Iteratively apply *genetic operators mutation*, *recombination*, *selection* to a population of candidate solutions.

Inspired by simple model of biological evolution:

- *Mutation* introduces random variation in the genetic material of individuals.
- *Recombination* of genetic material during sexual reproduction produces *offspring* that combines features inherited from both *parents*.
- Differences in *evolutionary fitness* lead *selection* of genetic traits (‘survival of the fittest’).
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Evolutionary Algorithm (EA):

determine initial population $sp$

While termination criterion is not satisfied:

generate set $spr$ of new candidate solutions by recombination

generate set $spm$ of new candidate solutions from $spr$ and $sp$ by mutation

select new population $sp$ from candidate solutions in $sp$, $spr$, and $spm$
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**Solution:** Apply subsidiary local search after initialisation, mutation and recombination.

⇒ *Memetic Algorithms* (aka *Genetic Local Search*)
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determine initial population \( sp \)

perform *subsidiary local search* on \( sp \)

While *termination criterion* is not satisfied:

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Memetic Algorithm (MA):

determine initial population $sp$

perform subsidiary local search on $sp$

While termination criterion is not satisfied:

| generate set $spr$ of new candidate solutions
| by recombination

| perform subsidiary local search on $spr$

| generate set $spm$ of new candidate solutions
| from $spr$ and $sp$ by mutation

| perform subsidiary local search on $spm$

| select new population $sp$ from
| candidate solutions in $sp$, $spr$, and $spm$
Initialisation

- *Often*: independent, uninformed random picking from given search space.
- *But*: can also use multiple runs of construction heuristic.

Recombination

- Typically repeatedly selects a set of *parents* from current population and generates *offspring* candidate solutions from these by means of *recombination operator*.
- *Recombination operators* are generally based on *linear representation* of candidate solutions and piece together *offspring* from fragments of *parents*.
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Example: One-point binary crossover operator

Given two parent candidate solutions $x_1x_2\ldots x_n$ and $y_1y_2\ldots y_n$:

1. choose index $i$ from set $\{2, \ldots, n\}$ uniformly at random;

2. define offspring as $x_1\ldots x_{i-1}y_i\ldots y_n$ and $y_1\ldots y_{i-1}x_i\ldots x_n$. 
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```
0 1 1 0 1 1 1 1 0
1 0 0 0 1 0 1 1 0
0 1 1 0 1 0 1 1 0
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```
Mutation

- **Goal**: Introduce relatively small perturbations in candidate solutions in current population + offspring obtained from recombination.

- Typically, perturbations are applied stochastically and independently to each candidate solution; amount of perturbation is controlled by *mutation rate*.

- Can also use *subsidiary selection function* to determine subset of candidate solutions to which mutation is applied.

- In the past, the role of mutation (as compared to recombination) in high-performance evolutionary algorithms has been often underestimated [Bäck, 1996].
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Selection (1)

- Determines population for next cycle (generation) of the algorithm by selecting individual candidate solutions from current population + new candidate solutions obtained from recombination, mutation (+ subsidiary local search).

- **Goal:** Obtain population of high-quality solutions while maintaining population diversity.

- Selection is based on evaluation function (fitness) of candidate solutions such that better candidate solutions have a higher chance of ‘surviving’ the selection process.
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Selection (2)

- Many selection schemes involve probabilistic choices, e.g., *roulette wheel selection*, where the probability of selecting any candidate solution \( s \) is proportional to its fitness value, \( g(s) \).

- It is often beneficial to use *elitist selection strategies*, which ensure that the best candidate solutions are always selected.

Subsidiary local search

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▶ **Search space:** set of all truth assignments for propositional variables in given CNF formula $F$; **solution set:** models of $F$; use **1-flip neighbourhood relation**; **evaluation function:** number of unsatisfied clauses in $F$.

▶ **Note:** truth assignments can be naturally represented as bit strings.

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- **Recombination:** Add offspring from \( n/2 \) (independent) one-point binary crossovers on pairs of randomly selected assignments from population to current population \((n = \text{number of variables in } F)\).

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- **Genetic Algorithms (GAs)** [Holland, 1975; Goldberg, 1989]:
  - have been applied to a very broad range of (mostly discrete) combinatorial problems;
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