Empirical Analysis of SLS Algorithms

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Outline

1. Las Vegas Algorithms
2. Run-Time Distributions
3. RTD-Based Analysis of LVA Behaviour
4. Characterising and Improving LVA Behaviour
SLS algorithms are typically *incomplete*: there is no guarantee that an (optimal) solution for a given problem instance will eventually be found.
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**But:** For decision problems, any solution returned is guaranteed to be correct.

**Also:** The run-time required for finding a solution (in case one is found) is subject to random variation.

⇝ These properties define the class of *(generalised)* Las Vegas algorithms, of which SLS algorithms are a subset.
Definition: (Generalised) Las Vegas Algorithm (LVA)

An algorithm \( A \) for a problem class \( \Pi \) is a (generalised) Las Vegas algorithm (LVA) iff it has the following properties:

1. If for a given problem instance \( \pi \in \Pi \), algorithm \( A \) terminates returning a solution \( s \), \( s \) is guaranteed to be a correct solution of \( \pi \).

Note: This is a slight generalisation of the definition of a Las Vegas algorithm known from theoretical computing science (our definition includes algorithms that are not guaranteed to return a solution).
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- Any SLS algorithm for a decision problem is also a Las Vegas algorithm. (Condition 1 is trivially satisfied because solutions are checked to be correct before they are returned.)
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- Las Vegas algorithms can be deterministic, since deterministic run-time is modelled by a degenerate probability distribution (aka Dirac delta distribution).
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1. $A$ is a (generalised) Las Vegas algorithm.

2. For any given instance $\pi' \in \Pi'$, the solution quality achieved by $A$ applied to $\pi'$ after any given run-time $t$ is a random variable $SQ_{A,\pi'}(t)$. 
Note:

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LVAs can be seen as special cases of Monte Carlo Algorithms, i.e., randomised algorithms that can sometimes return an incorrect solution to the given problem instance (false positive result).
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  - apply to worst-case or *highly idealised average-case behaviour* only;
  
  - capture only *asymptotic behaviour* and do not reflect *actual behaviour* with sufficient accuracy.
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- While not satisfied with model (and deadline not exceeded):
  1. design **computational experiment** to test model
  2. conduct computational experiment
  3. analyse experimental results
  4. revise model based on results
Asymptotic run-time behaviour of LVAs

- **completeness:**
  for each soluble problem instance $\pi$ there is a time bound $t_{max}(\pi)$ for the time required to find a solution.

- **probabilistic approximate completeness (PAC property):**
  for each soluble problem instance a solution is found with probability $\rightarrow 1$ as run-time $\rightarrow \infty$.

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- \textit{essential incompleteness:}
  for some soluble problem instances, the probability for finding a solution is strictly smaller than 1 for run-time \( \rightarrow \infty \).
Examples:

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- *Iterative Improvement* is essentially incomplete.
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  In many cases, these can render algorithms provably PAC; but effectiveness in practice can vary widely.
Asymptotic behaviour of OLVAs

- Simple generalisation based on associated decision problems for given solution quality bound $q := r \cdot q^*$, where $q^*$ = optimal solution quality for given problem instance:
  
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- Terminology for optimal solution qualities:

  - complete $= 1$-complete, PAC $= 1$-PAC, essentially incomplete $= \text{essentially } 1$-incomplete.
Application scenarios and evaluation criteria (1)

Evaluation criteria for LVAs depend on the application context:

- **Type 1:** No time limits given, algorithm can be run until a solution is found (off-line computations, non-realtime environments, e.g., configuration of production facility).

  ⇝ evaluation criterion: expected run-time
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  \[ \text{evaluation criterion: expected run-time} \]

- **Type 2:** Hard time limit \( t_{max} \) for finding solution; solutions found later are useless (real-time environments with strict deadlines, e.g., dynamic task scheduling or on-line robot control).
  \[ \text{evaluation criterion: solution probability at time } t_{max} \]
Application scenarios and evaluation criteria (2)

In many real applications, utility of solutions depends in more complex ways on time required for finding them:

- **Type 3:** Characterised by utility function $U : \mathbb{R}^+ \mapsto [0, 1]$, where $U(t) =$ utility of solution found at time $t$. 

Example: Direct benefit of solution is invariant over time, but cost of compute time diminishes final payoff according to

$$U(t) := \max \{u_0 - c \cdot t, 0\}$$

(constant discounting).

Evaluation criterion for type 3 scenario: utility-weighted solution probability

$$U(t) \cdot P_{s(\text{RT} \leq t)} \Rightarrow \text{requires detailed knowledge of } P_{s(\text{RT} \leq t)} \text{ for arbitrary } t.$$
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**Some scenarios:**

- Run-time is unconstrained, given solution quality threshold must be reached (generalisation of type 1 scenario)

- Hard time-limit is given, during which best possible solution quality should be found (generalisation of type 2 scenario).
In many cases, tradeoffs between run-time and solution quality are more complex.

- **Generalisation of type 3 scenario:** Utility of solution depends on quality and time needed for finding it; characterised by utility function $U : \mathbb{R}^+ \times \mathbb{R}^+ \mapsto [0, 1]$, where $U(t, q) = \text{utility of solution of quality } q \text{ found at time } t$. 
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  $\sim$ requires detailed knowledge of $P_s(\text{RT} \leq t, SQ \leq q)$. 
Las Vegas algorithms are often designed and evaluated without a priori knowledge of the application scenario; therefore:

- Assume most general scenario: type 3 with unknown utility function;
- Evaluate based on solution probabilities $P_{s}(RT \leq t)$ or $P_{s}(RT \leq t, SQ \leq q)$ for arbitrary run-times $t$ and solution qualities $q$.

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study distributions of \textit{random variables} characterising run-time and solution quality of algorithm on given problem instance.
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Definition: Run-Time Distribution (1)

Given Las Vegas algorithm $A$ for decision problem $\Pi$:

- The success probability $P_s(\text{RT}_A, \pi \leq t)$ is the probability that $A$ finds a solution for a soluble instance $\pi \in \Pi$ in time $\leq t$. 
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- The *run-time distribution (RTD) of $A$ on $\pi$* is the probability distribution of the random variable $\text{RT}_A,\pi$. 

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- The *run-time distribution function rtd* : $\mathbb{R}^+ \mapsto [0, 1]$, defined as $rtd(t) = P_s(RT_{A,\pi} \leq t)$, completely characterises the RTD of $A$ on $\pi$. 
Definition: **Run-Time Distribution (2)**

Given OLVA $A'$ for optimisation problem $\Pi'$:

- The *success probability* $P_s(\mathit{RT}_{A',\pi'} \leq t, \mathit{SQ}_{A',\pi'} \leq q)$ is the probability that $A'$ finds a solution for a soluble instance $\pi' \in \Pi'$ of quality $\leq q$ in time $\leq t$. 

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- The *run-time distribution (RTD) of $A'$ on $\pi'$* is the probability distribution of the bivariate random variable $(RT_{A',\pi'}, SQ_{A',\pi'})$.
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- The *run-time distribution (RTD) of $A'$ on $\pi'$* is the probability distribution of the bivariate random variable $(RT_{A'},\pi', SQ_{A'},\pi')$.

- The *run-time distribution function* $rtd : \mathbb{R}^+ \times \mathbb{R}^+ \mapsto [0, 1]$, defined as $rtd(t, q) = P_s(RT_{A,\pi} \leq t, SQ_{A',\pi'} \leq q)$, completely characterises the RTD of $A'$ on $\pi'$. 
Typical run-time distribution for SLS algorithm applied to hard instance of combinatorial optimisation problem:

\[
P(\text{solve}) \quad \text{rel. soln. quality [%]} \quad \text{run-time [CPU sec]}
\]

0.80.60.40.2

0

2.5

2

1.5

1

0.5

0

0.1

1

10

100

P(solve)

rel. soln. quality [%]

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Qualified RTDs for various solution qualities:

![Graph showing the relationship between relative solution quality and run-time](image-url)
Qualified run-time distributions (QRTDs)

- A qualified run-time distribution (QRTD) of an OLVA $A'$ applied to a given problem instance $\pi'$ for solution quality $q'$ is a marginal distribution of the bivariate RTD $rtd(t, q)$ defined by:

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- QRTDs characterise the ability of a given SLS algorithm for a combinatorial optimisation problem to solve the associated decision problems.

Note: Solution qualities $q$ are often expressed as relative solution qualities $q/q^* - 1$, where $q^*$ = optimal solution quality for given problem instance.
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Solution quality distributions for various run-times:
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- SQDs correspond to cross-sections of the two-dimensional bivariate RTD graph.

- SQDs characterise the solution qualities achieved by a given SLS algorithm for a combinatorial optimisation problem within a given run-time bound (useful for type 2 application scenarios).
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- For PAC algorithms, the SQDs for very large time-limits $t'$ approach degenerate distributions that concentrate all probability on the optimal solution quality.

- For any essentially incomplete algorithm $A'$ (such as Iterative Improvement) applied to a problem instance $\pi'$, the SQDs for sufficiently large time-limits $t'$ approach a non-degenerate distribution called the \textit{asymptotic SQD of $A'$ on $\pi'$}.
Solution quality statistics over time (SQTs)

- The development of solution quality over the run-time of a given OLVA is reflected in time-dependent SQD statistics (solution quality over time (SQT) curves).

- Important aspects of an algorithm's run-time behaviour may be easily missed when basing an analysis solely on a single SQT curve.
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- SQT curves are widely used to illustrate the trade-off between run-time and solution quality for a given OLVA.
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- **But:** Important aspects of an algorithm’s run-time behaviour may be easily missed when basing an analysis solely on a single SQT curve.
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Empirically measuring RTDs

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- Higher numbers of runs (larger sample sizes) give more accurate approximations of a true RTD.
Typical sample of run-times for an SLS algorithm applied to an instance of a hard decision problem:
Corresponding empirical RTD:
Protocol for obtaining the empirical RTD for an LVA $A$ applied to a given instance $\pi$ of a decision problem:

- Perform $k$ independent runs of $A$ on $\pi$ with cutoff time $t'$. (For most purposes, $k$ should be at least 50–100, and $t'$ should be high enough to obtain at least a large fraction of successful runs.)
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▶ Sort $L$ according to increasing run-time; let $rt(j)$ denote the run-time from entry $j$ of the sorted list ($j = 1, \ldots, k'$).

▶ Plot the graph $(rt(j), j/k)$, i.e., the cumulative empirical RTD of $A$ on $\pi$. 
The fraction of successful runs, $sr := k'/k$, is called the \textit{success ratio}; for large run-times $t'$, it approximates the \textit{asymptotic success probability} $p_s^* := \lim_{t \to \infty} P_s(RT_{a,\pi} \leq t)$. 

$\frac{1}{sr} - 1$ is the expected number of failed runs required before a successful run is observed.
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The mean run-time for a variant of the algorithm that restarts after time \( t' \) can be estimated as:

\[
\hat{E}(RT_s) + (1/sr - 1) \cdot \hat{E}(RT_f)
\]

where \( \hat{E}(RT_s) \) and \( \hat{E}(RT_f) \) are the average times of successful and failed runs, respectively.

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- During each run, whenever the incumbent solution is improved, record the quality of the improved incumbent solution and the time at which the improvement was achieved in a solution quality trace.
- Let $sq(t', j)$ denote the best solution quality encountered in run $j$ up to time $t'$. The cumulative empirical RTD of $A'$ on $\pi'$ is defined by $\hat{P}_s(RT \leq t', SQ \leq q') := \#\{j \mid sq(t', j) \leq q'\}/k$.

Note: Qualified RTDs, SQDs and SQT curves can be easily derived from the same solution quality traces.
Measuring run-times (1):

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When reporting CPU times, the run-time environment should be specified (at least CPU type, model, speed and cache size; amount of RAM; OS type and version); ideally, the implementation of the algorithm should be made available.
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- *cost models* that specify the CPU time for each such operation for a given implementation and run-time environment.
Example:

For a given SLS algorithm for SAT applied to a specific SAT instance we observe

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when running the algorithm on an Intel Xeon 2.4GHz CPU with 512KB cache and 1GB RAM running Red Hat Linux, Version 2.4smp (*run-time environment*).
Run-length distributions:

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- Elementary operations commonly used as the basis for RLD and other run-time measurements of SLS algorithms include search steps, objective function evaluations and updates of data structures used for implementing the step function.
RTD-based Analysis of LVA Behaviour

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RTD-based empirical analysis in combination with proper statistical techniques (hypothesis tests) is a state-of-the-art approach in empirical algorithmics.
**RTD plots** are useful for the *qualitative analysis* of LVA behaviour:

- *Semi-log plots* give a better view of the distribution over its entire range.
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- Uniform performance differences characterised by a constant factor correspond to shifts along horizontal axis.

- **Log-log plots** of an RTD or its associated failure rate decay function, $1 – rtd(t)$, are often useful for examining behaviour for very short or very long runs.
Various graphical representations of a typical RTD:
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Quantitative RTD analysis is typically based on basic descriptive statistics, such as:

- mean;
- median ($q_{0.5}$) and other quantiles (e.g., $q_{0.25}$, $q_{0.75}$, $q_{0.9}$);
- standard deviation or (better) variation coefficient $vc = \frac{\text{stddev}}{\text{mean}}$;
- quantile ratios, such as $q_{0.75}/q_{0.5}$ or $q_{0.9}/q_{0.1}$.

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The empirical RLD of a given SLS algorithm for SAT on a specific SAT instance is characterised by the following basic descriptive statistics:

<table>
<thead>
<tr>
<th>Statistic</th>
<th>Value</th>
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<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>mean</td>
<td>57 606.23</td>
<td>median</td>
<td>38 911</td>
</tr>
<tr>
<td>min</td>
<td>107</td>
<td>q_{0.25}</td>
<td>16 762; 5 332</td>
</tr>
<tr>
<td>max</td>
<td>443 496</td>
<td>q_{0.75}</td>
<td>80 709; 137 863</td>
</tr>
<tr>
<td>stddev</td>
<td>58 953.60</td>
<td>q_{0.75}/q_{0.25}</td>
<td>4.81</td>
</tr>
<tr>
<td>vc</td>
<td>1.02</td>
<td>q_{0.9}/q_{0.1}</td>
<td>25.86</td>
</tr>
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Note:

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- Unlike the standard deviation (or variance), the variation coefficient and quantile ratios are invariant under multiplication by constants.
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- QRTDs and SQDs can be handled analogously to RTDs; along with SQT curves they can be easily determined from the same solution quality traces that provide the basis for empirical bivariate RTDs of a given optimisation LVA.
Basic quantitative analysis for ensembles of instances (1)

- In principle, the same approach as for individual instances is applicable: Measure empirical RTD for each instance, analyse using RTD plots or descriptive statistics.
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\[\rightsquigarrow\text{ Select typical instance for presentation or further analysis, briefly summarise data for remaining instances.}\]
RTDs for WalkSAT/SKC, a prominent SLS algorithm for SAT, on three hard 3-SAT instances:
Basic quantitative analysis for ensembles of instances (2)

- For bigger sets of instances (e.g., samples from random instance distributions), it is important to characterise the performance of the given algorithm on individual instances as well as across the entire ensemble.

Useful fact: Exponential and polynomial functions appear as straight lines in semi-log plots and log-log plots, respectively.
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Distribution of median search cost for WalkSAT/ SKC over set of 1000 randomly generated, hard 3-SAT instances:
Some criteria for constructing/selecting benchmark sets:

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- For an instance of an optimisation problem, OLVA A' probabilistically dominates OLVA B' on a given problem instance iff for all solution quality bounds, A' probabilistically dominates B' on the respective associated decision problem.
Comparing algorithms based on RTDs (2)

- A probabilistic domination relation holds between two Las Vegas algorithms on a given problem instance iff their respective (qualified) RTDs do not cross each other. Even for single problem instances, a probabilistic domination relation does not always hold (i.e., there is a cross-over between the respective RTDs). In this situation, which of two given algorithms is superior depends on the time both algorithms are allowed to run.

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Background: Statistical hypothesis tests (1)

- **Statistical hypothesis tests** are used to assess the validity of statements about properties of or relations between sets of statistical data.

- The statement to be tested (or its negation) is called the null hypothesis ($H_0$) of the test. Example: For the Mann-Whitney U-test, the null hypothesis is 'the two distributions underlying two given samples have the same median'.

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- Most common statistical hypothesis tests and other statistical analyses can be performed rather conveniently in the free *R software environment* (see http://www.r-project.org/).
Comparing algorithms based on RTDs (3)

- The **Mann Whitney U-test** (aka *Wilcoxon rank sum test*) is used to test whether the medians of two samples (e.g., empirical RTDs) are significantly different.

Unlike the widely used *t*-test, the U-test is distribution-free (or non-parametric), i.e., it does not depend on the assumption that the underlying probability distributions are Gaussian. (This assumption is typically violated for the RTDs of SLS algorithms.)

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Performance differences detectable by the Mann-Whitney U-test for various sample sizes (sign. level 0.05, power 0.95):

<table>
<thead>
<tr>
<th>sample size</th>
<th>$m_1/m_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>3 010</td>
<td>1.1</td>
</tr>
<tr>
<td>1 000</td>
<td>1.18</td>
</tr>
<tr>
<td>122</td>
<td>1.5</td>
</tr>
<tr>
<td>100</td>
<td>1.6</td>
</tr>
<tr>
<td>32</td>
<td>2</td>
</tr>
<tr>
<td>10</td>
<td>3</td>
</tr>
</tbody>
</table>

$m_1/m_2$ is the ratio between the medians of the two empirical distributions.
Example of crossing RTDs for two SLS algorithms for the TSP applied to a standard benchmark instance (1000 runs/RTD):
Comparative analysis for instance ensembles (1)

**Goal:** Compare performance of Las Vegas algorithms $A$ and $B$ on a given ensemble of instances.

- Use instance-based analysis to partition given ensemble into three subsets:
  - instances on which $A$ probabilistically dominates $B$;
  - instances on which $B$ probabilistically dominates $A$;
  - instances on which there is no probabilistic domination between $A$ and $B$ (crossing RTDs).

The size of these subsets gives a rather detailed picture of the algorithms' relative performance on the given ensemble.
Comparative analysis for instance ensembles (1)

**Goal:** Compare performance of Las Vegas algorithms $A$ and $B$ on a given ensemble of instances.

- Use instance-based analysis to partition given ensemble into three subsets:
  - instances on which $A$ probabilistically dominates $B$;
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- Use statistical tests to assess significance of performance differences across given instance ensemble.

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- The *binomial sign test* measures whether the median of the paired performance differences (e.g., in median run-time) of $A$ and $B$ per instance is significantly different from zero, which means that there is no significant *systematic* performance difference between $A$ and $B$ across the ensemble.

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- For qualitative correlation analyses, scatter plots in which each instance $\pi$ is represented by one point whose $x$ and $y$ co-ordinates correspond to the performance of $A$ and $B$ on $\pi$. 
Correlation between median run-time for two SLS algorithms for the TSP over a set of 100 randomly generated instances:

10 runs per instance.
Comparative analysis for instance ensembles (4)

- Quantitatively, the correlation can be summarised using the *empirical correlation coefficient*. Additionally, *regression analysis* can be used to model regular performance relationships.
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- To test the statistical significance of an observed *monotonic* relationship, use non-parametric tests such as *Spearman’s rank order test*. 
Correlation between median run-time for two SLS algorithms for the TSP over a set of 100 randomly generated instances:

10 runs per instance; correlation coefficient 0.39, significant according to Spearman’s rank order test at $\alpha = 0.05$; p-value $= 9 \cdot 10^{-11}$. 
Most high-performance SLS algorithms have parameters that significantly affect their performance.

Note: Peak performance is a measure of potential performance.

Pitfall: Unfair parameter tuning, i.e., the use of unevenly optimised parameter settings in comparative studies.
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- **Note:**
  - Optimal parameter settings often vary substantially between problem instances or instance classes.
  - Effects of multiple parameters are typically *not* independent.

- *Performance robustness*, i.e., the variation in performance due to deviations from optimal parameter settings, is an important performance criterion.
Peak Performance vs Robustness (3)

- Performance robustness can be studied empirically by measuring the impact of parameter settings on RTDs (or their descriptive statistics) of a given LVA on a set of problem instances.
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- Advanced empirical studies should attempt to relate the latter type of variations to features of the respective instances or domains (e.g., scaling studies relate LVA performance to instance size).
Characterising and Improving LVA Behaviour

Advanced aspects of empirical analysis include:

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▶ designing or configuring *restart strategies* and other *diversification mechanisms*,
▶ realising speedups through *multiple independent runs parallelisation*. 
Asymptotic behaviour and stagnation

The three previously discussed norms of LVA behaviour, *completeness, PAC property* and *essential incompleteness*, correspond to properties of an algorithm’s theoretical RTDs.
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- **Completeness** can be empirically falsified for a given time-bound, but it cannot empirically verified.

- Neither the **PAC property**, nor **essential incompleteness** can be empirically verified or falsified.

- **But:** Empirical RTDs can provide evidence (rather than proof) for essential incompleteness or PAC behaviour.
Example of asymptotic behaviour in empirical RTDs:

Note: $MMAS$ is provably PAC, $MMAS^*$ is essentially incomplete.
LVA efficiency and stagnation

- In practice, the rate of decrease in the failure probability, $\lambda_{A,\pi}(t)$, is more relevant than true asymptotic behaviour.

Note: Exponential RTDs are characterised by a constant rate of decrease in failure probability. A drop in $\lambda_{A,\pi}(t)$ indicates stagnation of algorithm A’s progress towards finding a solution of instance $\pi$. Stagnation can be detected by comparing the RTD against an exponential distribution.
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- Stagnation can be detected by comparing the RTD against an exponential distribution.
Evidence of stagnation in an empirical RTD:

‘ed[18]’ is the CDF of an exponential distribution with median 18; the arrows mark the point at which stagnation behaviour becomes apparent.
Note:

- The formal definition of LVA *efficiency* and *stagnation* is based on the idea that an LVA $A$ suffers from stagnation iff its success probability can be increased by restarting $A$ after an appropriately chosen cutoff time.

(For details, see Definition 4.9 on page 187 of SLS:FA.)
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- Efficiency and stagnation are *relative measures*; they cannot indicate all situations in which a given LVA’s behaviour can be further improved.
Functional characterisation of LVA behaviour (1)

- Empirical RTDs are step functions that approximate the underlying theoretical RTDs.
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- Approximations with parameterised families of continuous distribution functions known from statistics, such as exponential or normal distributions, are particularly useful.
Approximation of an empirical RTD with an exponential distribution $ed[m](x) := 1 - 2^{-x/m}$.
Functional characterisation of LVA behaviour (2)

- **Model fitting techniques**, such as the *Marquardt-Levenberg* or *Expectation Maximisation algorithms*, can be used to find good approximations of empirical RTDs with parameterised cumulative distribution functions.
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- The quality of approximations can be assessed using *statistical goodness-of-fit tests*, such as the $\chi^2$-test or the *Kolmogorov-Smirnov test*. 


Note: Particularly for small or easy problem instances, the quality of optimal functional approximations can sometimes be limited by the inherently discrete nature of empirical RTD data.

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Approximation of an empirical RTD with an exponential distribution $\text{ed}[m](x) := 1 - 2^{-x/m}$:

The optimal fit exponential distribution obtained from the Marquardt-Levenberg algorithm passes the $\chi^2$ goodness-of-fit test at $\alpha = 0.05$. 
Performance improvements based on static restarts (1)

- Detailed RTD analyses can often suggest ways of improving the performance of a given SLS algorithm.
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- A static restart strategy is effective, i.e., leads to increased solution probability for some run-time $t''$, if the RTD of the given algorithm and problem instance is less steep than an exponential distribution crossing the RTD at some time $t < t''$. 
Example of an empirical RTD of an SLS algorithm on a problem instance for which static restarting is effective:

‘ed[18]’ is the CDF of an exponential distribution with median 18; the arrows mark the optimal cutoff-time for static restarting.
Performance improvements based on static restarts (2)

- To determine the optimal cutoff-time $t_{opt}$ for static restarts, consider the left-most exponential distribution that touches the given empirical RTD and choose $t_{opt}$ to be the smallest $t$ value at which the two respective distribution curves meet.

  (For a formal derivation of $t_{opt}$, see page 193 of SLS:FA.)
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- **Note:** This method for determining optimal cutoff-times only works *a posteriori*, given an empirical RTD.
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- Optimal cutoff-times for static restarting typically vary considerably between problem instances; for optimisation algorithms, they also depend on the desired solution quality.
Overcoming stagnation using dynamic restarts

- Dynamic restart strategies are based on the idea of re-initialising the search process only when needed, i.e., when stagnation occurs.
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- **Simple dynamic restart strategy**: Re-initialise search when the time interval since the last improvement of the incumbent candidate solution exceeds a given threshold $\theta$. (Incumbent candidate solutions are not carried over restarts.)
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Example: Effect of simple dynamic restart strategy

![Graph showing the effect of simple dynamic restart strategy](image-url)
Other diversification strategies

- Restart strategies often suffer from the fact that search initialisation can be relatively time-consuming (setup time, time required for reaching promising regions of given search space).

- Effective techniques for overcoming search stagnation are crucial components of high-performance SLS methods.

- This problem can be avoided by using other diversification mechanisms for overcoming search stagnation, such as random walk extensions that render a given SLS algorithm provably PAC; adaptive modification of parameters controlling the amount of search diversification, such as temperature in SA or tabu tenure in TS.
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Multiple independent runs parallelisation

- Any LVA $A$ can be easily parallelised by performing multiple runs on the same problem instance $\pi$ in parallel on $p$ processors.

The effectiveness of this approach depends on the RTD of $A$ on $\pi$: Optimal parallelisation speedup of $p$ is achieved for an exponential RTD. The RTDs of many high-performance SLS algorithms are well approximated by exponential distributions; however, deviations for short run-times (due to the effects of search initialisation) limit the maximal number of processors for which optimal speedup can be achieved in practice.
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Speedup achieved by multiple independent runs parallelisation of a high-performance SLS algorithm for SAT: